Multiple forms of intermittency in partial differential equation dynamo models

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We find concrete evidence for the presence of crisis-induced and Pomeau-Manneville type-I intermittencies in an axisymmetric partial differential equation (PDE) mean-field dynamo model. These findings are of potential importance for two different reasons. First, as far as we are aware, this is the first time detailed evidence has been produced for the occurrence of these types of intermittency for such deterministic PDE models. And second, despite the rather idealized nature of these models, the concrete evidence for the occurrence of more than one type of intermittency in such models makes it in principle possible that different types of intermittency may occur in different solar-type stars or even in the same star over different epochs. In this way a *multiple intermittency framework* may turn out to be of importance in understanding the mechanisms responsible for grand-minima type behavior in the Sun and solar-type stars and in particular in the interpretation of the corresponding observational and proxy evidence. [S1063-651X(99)08811-X]

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I. INTRODUCTION

Intermittency has been observed in a variety of real settings as well in a vast number of numerical models. A great deal of effort has therefore gone into understanding these modes of behavior in the context of deterministic dynamical systems theory. These studies have demonstrated the existence of a number of different types of intermittency (such as Pomeau-Manneville [1], crisis [2], and on-off [3] intermittencies), each with their own associated signatures and scalings. Many of these forms of intermittency have in turn been concretely shown to be present in experiments and numerical studies of dynamical systems in a variety of settings (see [4–6] and references therein).

An important potential domain of applicability of such behavior arises in understanding the mechanisms underlying the intermediate time-scale variability in the Sun [7]—the occurrence of the so-called *maunder* or *grand minima*—during which solar activity (as deduced from the sunspot numbers) was greatly diminished [7,8]. This behavior is also confirmed by evidence coming from the analysis of proxy data [9]. There is also some evidence for similar types of variability in solar-type stars [10].

The idea that some type of dynamical intermittency may underpin the grand minima type variability in the sunspot record (the *intermittency hypothesis* [11]) goes back at least to the late 1970s [12–14]. This idea has been the subject of intense study over the recent years and has involved the employment of various classes of dynamo models, including ordinary differential equation (ODE) (e.g., [13,15]) as well as partial differential equation (PDE) models (e.g., [16–18,24]). In addition to the phenomenological evidence for the presence of intermittent-type behaviors in dynamo models [16–20], concrete evidence has recently been found for the presence of particular types of intermittency in both ODE dynamo models [21,22] as well as a recently discovered generalization of on-off intermittency, referred to as *in-out* intermittency [23], in PDE models [24].

Here we wish to report concrete evidence for the occurrence of two other types of intermittency, namely the crisis-induced and Pomeau-Manneville type-I intermittencies, in PDE mean-field dynamo models. The organization of the paper is as follows. In Sec. II we briefly introduce the model studied here. Section III summarizes our evidence demonstrating the presence of these types of intermittencies in this model, and finally in Sec. IV we draw our conclusions.

II. MODEL

Ideally one would wish to employ the full three-dimensional (3D) dynamo models with a minimum number of approximations and simplifying assumptions. Despite a number of important recent attempts [25–27], the difficulty of dealing with small scale turbulence makes a detailed and extensive self-consistent study of such fully turbulent regimes in stars still computationally impractical (see, e.g., [26,28–30]).

In view of this, an alternative approach in studies of stellar dynamos has been to employ mean-field models [15,16,18,19,31,32]. We should mention that there is an ongoing debate regarding the nature and realistic value of such models [30]. Nevertheless, 3D turbulence simulations do seem to produce magnetic fields whose global properties (such as field parity and time dependence) are similar to those expected from corresponding mean-field dynamo models [33]. In this way mean-field dynamo models seem to reproduce certain features of the more complicated models and allow the study of certain global properties of magnetic fields in the Sun and solar-type stars (see, for example, [33,34]).

The standard mean-field dynamo equation is given by

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B} + \alpha \mathbf{B} - \eta_t \nabla \times \mathbf{B}), \qquad (1)$$

where $\mathbf{B}$ and $\mathbf{u}$ are the mean magnetic field and mean velocity, respectively, and the turbulent magnetic diffusivity η_t

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and the coefficient α arise from the correlation of small scale turbulent velocities and magnetic fields (α effect) [35]. We consider the usual algebraic form of α quenching, namely

$$\alpha = \frac{\alpha_0 \cos\theta}{1+|\mathbf{B}|^2}, \tag{2}$$

where $\alpha_0 = \text{const}$ and θ is the co-latitude.

We solve Eq. (1) in an axisymmetric configuration, and in the following, as is customary [32], we shall discuss the behavior of the solutions by monitoring the total magnetic energy, $E = 1/2\mu_0 \int \mathbf{B}^2 dV$, where μ_0 is the induction constant, and the integral is taken over the dynamo region. We split E into two parts, $E = E_A + E_S$, where E_A and E_S are respectively the energies of the antisymmetric and symmetric parts of the field with respect to the equator. The overall parity P is given by $P = [E_S - E_A]/E$, so $P = -1$ denotes an antisymmetric (dipolelike) pure parity solution and $P = +1$ a symmetric (quadrupolelike) pure parity solution.

For the numerical results reported in the following section, we used a modified version of the axisymmetric dynamo code of Brandenburg *et al.* [32] employed recently in [36]. These models are constructed from a complete sphere of radius R by removing an inner concentric sphere of radius r_0 and a conical section of semiangle θ_0 about the rotation axis, from both the north and south polar regions (see [36] for details of the model and the relevant parameters). To test the robustness of the code we verified that no qualitative changes were produced by employing a finer grid and different temporal step length (we used a grid size of 41×81 mesh points and a step length of $10^{-4}R^2/\eta_t$ in the results presented in this paper). For the following results we use $C_\Omega = -10^4$, which give the magnitude of the differential rotation and $\theta_0 = 45°$. The magnitude of the α effect is given by the dynamo parameter C_α. In the next section we show in turn concrete evidence for the occurrence of crisis-induced and Pomeau-Manneville type-I intermittencies.

III. RESULTS

A. Crisis-induced intermittency

As far as their detailed underlying mechanisms and temporal signatures are concerned, crises come in three varieties [2]. Here we shall be concerned with only one of these types, referred to as an ''attractor merging crisis,'' whereby as a system parameter is varied, two or more chaotic attractors merge to form a single attractor. There is both experimental and numerical evidence for this type of intermittency (see, for example, [2,4] and references therein). In particular, this type of behavior has been discovered in a six-dimensional truncation of mean-field dynamo models [21]. Figure 1 shows the plots of the energy and parity for the above model as a function of time, calculated with $r_0 = 0.2$ and $C_\alpha = 25.202$, which show a bimodal behavior, switching intermittently between two different chaotic states.

To determine the nature of this behavior more precisely, we have plotted in Fig. 2 the return maps for the PDE models (1), showing the attractors before and after the merging. As can be seen, the resulting merged attractor is, as expected, larger than the superposition of the two preexisting attractors.

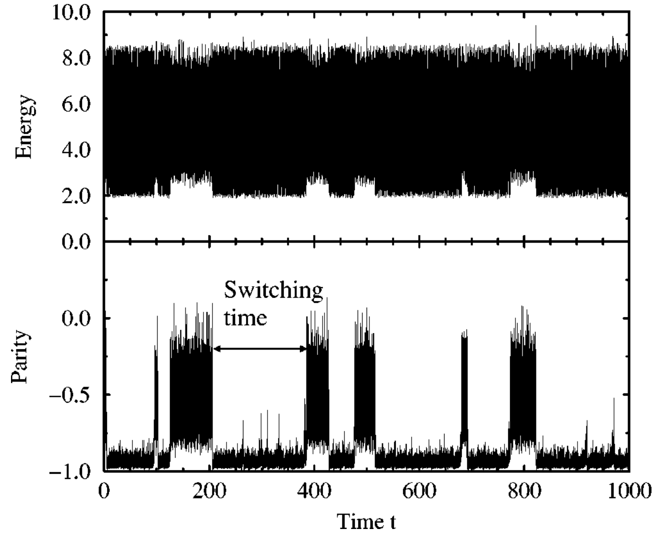

FIG. 1. Example of crisis-induced intermittency in a shell dynamo with a cut, with $r_0 = 0.2$, $C_\alpha = 25.202$, $C_\Omega = -10^4$, and $\theta_0 = 45°$.

These results can be taken as indications for the presence of crisis-induced intermittency in this model. To substantiate this further, we recall that another important signature of this type of intermittency is the way τ, the average time between switches, scales with the system parameter, in this case, C_α. According to Grebogi *et al.* [2], for a large class of dynamical systems this relation takes the form

$$\tau \sim |C_\alpha - C_\alpha^*|^{-\gamma}, \tag{3}$$

where the real constant γ is the critical exponent characteristic of the system under consideration and C_α^* is the critical value of C_α at which the two chaotic attractors merge.

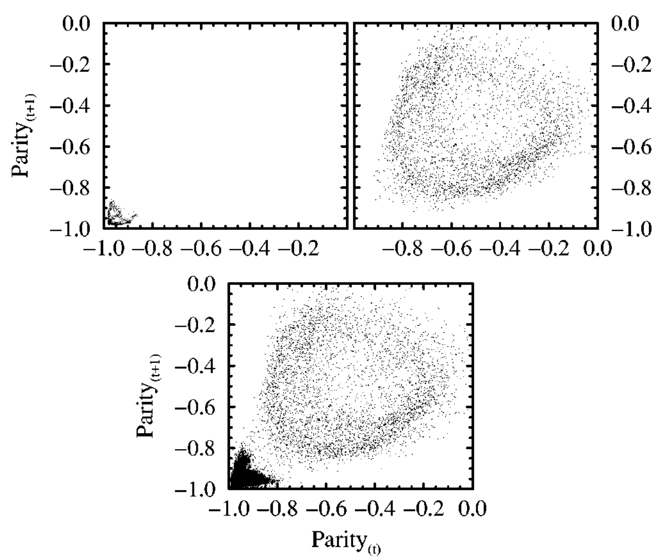

FIG. 2. Return maps showing the attractors in the PDE model (1) before (top panels) and after (bottom panel) the merging. Note that, as expected, the merged attractor is larger than the superposition of the two previous attractors.

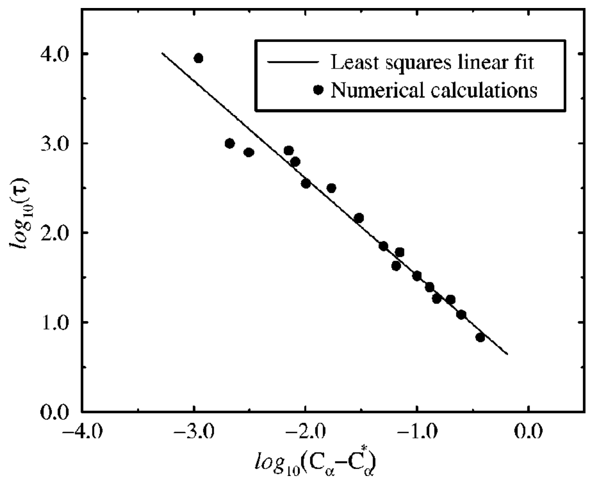

FIG. 3. Scaling of the average times between switches τ as a function of $(C_\alpha - C^*_\alpha)$ for crisis-induced intermittency for the model (1). The slope is found to be $\gamma = 1.08 \pm 0.05$.

The model under study here is a PDE system that is formally infinite dimensional. Such PDE models are numerically costly to integrate over long enough intervals of time (sometimes in excess of 5000 time units) necessary in order to obtain the scaling of the type (3). Furthermore, the demonstration of such scaling requires a precise determination of the critical value C^*_α, which is difficult since as one approaches this value τ diverges and the integration time becomes prohibitive. Despite these difficulties, we have succeeded in obtaining strong evidence for the presence of such a scaling as depicted in Fig. 3, with the corresponding $\gamma = 1.08 \pm 0.05$. Grebogi *et al.* [2] conjecture that there may be a general tendency for γ to be larger for higher-dimensional attractors. We do have a value of γ higher than the previous one found for a related six-dimensional ODE dynamo model [21] but much lower than the value range suggested by Grebogi *et al.* Therefore, the conjectured range may need modification for large high-dimensional systems.

There is also evidence for an enlargement of the final attractor after merging, as shown by the larger amplitudes of variation in the parity, in the sense that the parity gets closer to -1 after the merging, as depicted in Fig. 2. This helped us to numerically arrive at a better estimate for the critical value C^*_α. These indicators, taken together, amount to strong evidence for the presence of crisis-induced intermittency for this model.

B. Pomeau-Manneville type-I intermittency

This type of intermittency, which is brought about through a tangent bifurcation, results in the system switching back and forth between a ''ghost'' periodic orbit and sudden bursts of chaotic behavior [1]. There is both experimental and numerical evidence for this type of intermittency (see, for example, [5,37] and references therein). In particular, this type of behavior has been discovered in a 12-dimensional truncation of the mean-field dynamo model [22].

To demonstrate the presence of this type of intermittency in the above PDE dynamo model, we have plotted in Fig. 4 the energy and parity as a function of time for the parameter values $r_0 = 0.7$ and $C_\alpha = 28.0$, which clearly demonstrates switches between nearly periodic behavior and sudden bursts. We note that interestingly the energy in this case shows strong modulation that could be of interest in accounting for the occurrence of grand type minima in sunspot activity.

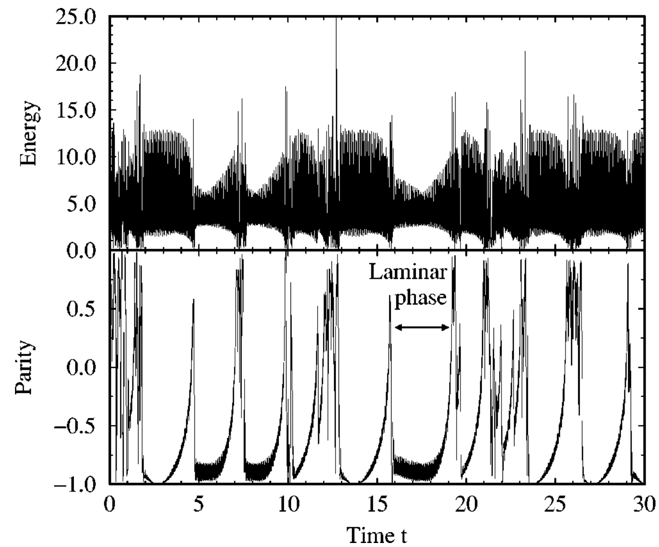

FIG. 4. Example of type-I intermittency in a shell dynamo with a cut, with $r_0 = 0.7$, $C_\alpha = 28.0$, $C_\Omega = -10^4$, and $\theta_0 = 45°$.

Another signature of this type of intermittency is provided by the specific characteristics of its corresponding power spectrum. By employing finite-dimensional maps [6], it has been shown that the corresponding spectra have a broadband feature whose shape obeys approximately the inverse-power law $1/f$ for $f > f_s$, where f_s is the saturation frequency. Below this frequency there is a flat plateau induced by noise that causes arbitrarily long laminar phases to become finite.

As further evidence for this type of intermittency in the model (1), we have plotted in Fig. 5 the power spectrum at $C_\alpha = 28.0$, obtained by averaging over 16 different initial conditions corresponding to different initial parities. As can be seen, the power spectrum shows both the flat plateau and

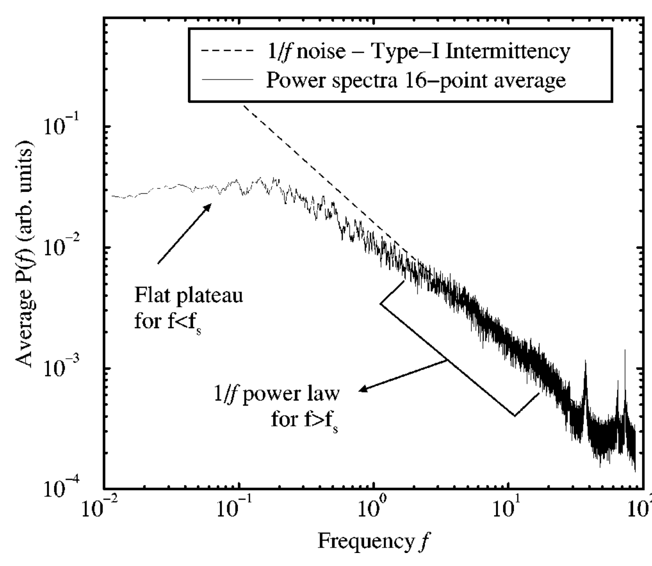

FIG. 5. Power spectra of the time series in Fig. 4 for type-I intermittency.

the 1/f power-law scaling. Taken together, these indicators amount to strong evidence for the presence of Pomeau-Manneville type-I intermittency for this model.

IV. CONCLUSION

We have obtained concrete evidence, in terms of phase space signatures, spectra, and scalings to demonstrate the presence of crisis-induced and the Pomeau-Manneville type-I intermittencies in axisymmetric mean-field PDE dynamo models. Despite the rather idealized nature of these models, this is of potential importance since it shows the occurrence of two more types of intermittency (in addition to in-out intermittency recently discovered [24]) in these models, which may in turn be taken as an indication that more than one type of intermittency may occur in solar and stellar dynamos. This suggests that any observational program for identifying the mechanisms underlying grand minima type variability needs to take into account the possibility that multiple intermittency mechanisms may be operative in different stars of similar type, or even in the same star over different epochs. This would also be of importance in the interpretation of proxy data. In this way a more appropriate hypothesis regarding such variability would be that of a *multiple-intermittency hypothesis*.

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